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A GENERAL MAXIMAL OPERATOR AND THE A_p-CONDITION

BY

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ABSTRACT. A rearrangement inequality for a general maximal operator $Mf(x) = \sup_{x \in Q} \int f\phi_Q \, d\nu$ is established. This is then applied to the Hardy-Littlewood maximal operator with weights.

1. Let μ, ν be two measures on $\mathbf{R}^n$ and let there be given for each cube $Q \subset \mathbf{R}^n$ a function ϕ_Q supported in Q. We consider the maximal operator $Mf(x) = \sup \int f\phi_Q \, d\nu$, where the sup is extended over all cubes centered at x and obtain (Theorem 1) the rearrangement inequality $(Mf)^*_\mu(\xi) \leq A\int_0^\infty \Phi(t) f^*_\nu(t\xi)\,dt$. Here g^*_λ denotes the nonincreasing rearrangement of g with respect to the measure λ, and Φ is a nonincreasing function given in terms of μ, ν, ϕ_Q. From this one easily sees that $\|Mf\|_{p,\mu} \leq A\|f\|_{p,\nu}\int_0^\infty \Phi(t)/t^{1/p}\,dt$, and thus the finiteness of this integral, i.e., $\Phi \in L(p', 1)$, gives a weighted norm inequality. This is how the A_p-condition comes into play. In fact, if we take $(u, v) \in A_p$, i.e., $\int_Q u \cdot (\int_Q v^{1-p'})^{p-1} \leq C|Q|^p$ [5], $d\mu = u\,dx$, $d\nu = v\,dx$, $\phi_Q(x) = \chi_Q(x)/|Q|v(x)$, then the above $Mf(x)$ is the usual Hardy-Littlewood maximal operator. Let $\Phi = \Phi_{u,v}$ be the associated Φ. We will show (Theorem 3) that, in the case $u = v$, $\Phi \in L(p', 1)$ if and only if $u \in A_p$, and in the double weight situation (Theorem 4), $\Phi \in L(p', \infty)$ if and only if there is $(\bar{u}, \bar{v})$ for which $\Phi_{\bar{u},\bar{v}} \sim \Phi$ and $\|Mf\|_{p,\bar{u}} \leq A\|f\|_{p,\bar{v}}$.

Finally, we will study the problem when $(u, v) \in A_p$ implies $\|Mf\|_{p,u} \leq A\|f\|_{p,v}$, and the extrapolation problem, i.e., when does $\|Mf\|_{p,u} \leq A\|f\|_{p,v}$ imply the existence of $\varepsilon > 0$ so that $\|Mf\|_{p-\varepsilon,u} \leq B\|f\|_{p-\varepsilon,v}$? It turns out that the behavior of the iterated maximal operator M_j is crucial here. We will see (Theorem 6) that extrapolation is possible provided the norm of M_j as an operator from $L^p_v \to L^p_u$ grows at most geometrically, a fact which is obvious for $u = v$. All this gives a different, though admittedly long, proof of $u \in A_p$, implies $u \in A_{p-\varepsilon}$, and shows that it is the iterated maximal operator that controls this implication.

2. For $\nu \geq 0$ a Borel measure on $\mathbf{R}^n$ and f: $\mathbf{R}^n \to \mathbf{R}$ a Borel measurable function, let $\lambda_{f,\nu}(y) = \nu\{x: |f(x)| > y\}$, and $f^*_\nu(t) = \inf\{y: \lambda_{f,\nu}(y) \leq t\}$, the rearrangement of f with respect to ν. With each $Q \in \{Q\}$, the collection of cubes in $\mathbf{R}^n$, let there be associated a Borel measurable function ϕ_Q: $\mathbf{R}^n \to [0, \infty)$, supp $\phi_Q \subset Q$. We consider the general maximal operator.

$$Mf(x) = \sup \int_{\mathbf{R}^n} \phi_Q f \, d\nu$$

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where the sup is extended over all Q with center x. If $\mu \geqslant 0$ is another Borel measure on $\mathbf{R}^n$, finite on compact sets, define

$$\Phi(t) = \sup_Q \left\{\mu(Q)\phi^*_{Q,\nu}(\mu(Q)t)\right\}.$$

THEOREM 1.

$$(Mf)^*_\mu(\xi) \leqslant A\int_0^\infty \Phi(t) f^*_\nu(t\xi)\,dt,$$

where A depends only upon the dimension n.

PROOF. We let $M_r f(x) = \sup \int \phi_Q f\,d\nu$, where now the sup is extended over all Q with center x and $\operatorname{diam} Q \leqslant r$. It suffices to prove the theorem for $M_r f$ and then let $r \to \infty$.

Let $E_\tau = \{x\colon M_r f(x) > \tau\}$ and $E_{\tau,R} = E_\tau \cap \{|x| \leqslant R\}$. For $x \in E_{\tau,R}$, we have a cube Q_x, center x, diam $Q_x \leqslant r$ such that $\tau \leqslant \int \phi_{Q_x} f\,d\nu$. We can now apply the Besicovitch covering theorem [1] and select $\{Q_j\} \subset \{Q_x\colon x \in E_{\tau,R}\}$ such that $E_{\tau,R} \subset \bigcup Q_j$ and $\Sigma \chi_{Q_j}(t) \leqslant C$, where C depends only upon n. Then $\mu(Q_j)\tau \leqslant \int \mu(Q_j)\phi_{Q_j} f\,d\nu$. We set $H_N = \Sigma_{j=1}^N \mu(Q_j)$, $\Phi_N(y) = \Sigma_{j=1}^N \mu(Q_j)\phi_{Q_j}(y)$. Then

$$H_N \leqslant \frac{1}{\tau}\int_{\mathbf{R}^n} \Phi_N(y) f(y)\,d\nu \leqslant \frac{1}{\tau}\int_0^\infty \Phi^*_{N,\nu}(t) f^*_\nu(t)\,dt.$$

We claim now that $\Phi^*_{N,\nu}(\xi) \leqslant c\Phi(\xi/H_N)$, where c is the Besicovitch constant. If $\Phi_N(y) > \alpha$, $\alpha > 0$, then $y \in \bigcup_{j=1}^N Q_j$. Thus the number of Q_j's containing y is at most c, and hence for some j, $\mu(Q_j)\phi_{Q_j}(y) > \alpha/c$. Thus

$$\{y\colon \Phi_N(y) > \alpha\} \subset \bigcup_{j=1}^N \left\{y\colon \mu(Q_j)\phi_{Q_j}(y) > \alpha/c\right\}.$$

We now show that for $\beta > 0$,

$$\nu\left\{y\colon \mu(Q)\phi_Q(y) > \beta\right\} \leqslant \mu(Q)\,|\{t\colon \Phi(t) > \beta\}|\,.$$

To prove this we may assume that $\mu(Q) > 0$. Then

$$\begin{aligned} |\{t\colon \Phi(t) > \beta\}| &\geqslant \left|\left\{t\colon \mu(Q)\phi^*_{Q,\nu}(\mu(Q)t) > \beta\right\}\right| \\ &= \frac{1}{\mu(Q)}\nu\left\{y\colon \mu(Q)\phi_Q(y) > \beta\right\}. \end{aligned}$$

All this gives us $\nu\{y\colon \Phi_N(y) > \alpha\} \leqslant H_N\,|\{t\colon \Phi(t) > \alpha/c\}|$. Consequently, $\Phi^*_{N,\nu}(\xi) \leqslant \inf\{\alpha\colon |\{t\colon \Phi(t) > \alpha/c\}| \leqslant \xi/H_N\} = c\Phi(\xi/H_N)$.

Thus

$$\tau \leqslant \frac{c}{H_N}\int_0^\infty \Phi\left(\frac{t}{H_N}\right) f^*_\nu(t)\,dt \leqslant \frac{c}{H_N}\int_0^\infty \Phi\left(\frac{t}{H}\right) f^*_\nu(t)\,dt,$$

since $H_N \leqslant H$, and $H = \Sigma\mu(Q_j) < \infty$. Since $H_N \uparrow H$, we get

$$\tau \leqslant \frac{c}{H}\int_0^\infty \Phi\left(\frac{t}{H}\right) f^*_\nu(t)\,dt = c\int_0^\infty \Phi(t) f^*_\nu(tH)\,dt,$$

and since $\mu(E_{\tau,R}) \leq H$, we see that $\tau \leq c\int_0^\infty \Phi(t) f_\nu^*(t\mu(E_{\tau,R}))\,dt$. Finally, let $\tau_0 = (M_r f)_\mu^*(\xi) = \inf\{\tau: \mu(E_\tau) \leq \xi\}$. Then $0 < \tau < \tau_0$ implies that $\mu(E_\tau) > \xi$, and hence for some R, $\mu(E_{\tau,R}) > \xi$. From this we get $\tau \leq c\int_0^\infty \Phi(t) f_\nu^*(t\xi)\,dt$, and letting $\tau \uparrow \tau_0$, completes the proof.

REMARK. Theorem 1 contains many of the known maximal inequalities.

(i) The choice $\phi_Q(y) = \chi_Q(y)/|Q|$, $\mu = \nu =$ Lebesgue measure, gives the ordinary Hardy-Littlewood maximal function. In this case $\Phi(t) = \chi_{[0,1]}(t)$ and so $(Mf)^*(\xi) \leq A\int_0^1 f^*(t\xi)\,dt$.

(ii) Let Q_0 be the unit cube centered at the origin, and let $Q(x, h)$ be the cube with center x, side-length h. Let $\operatorname{supp}\phi \subset Q_0$, and set $\phi_Q(y) = \phi((x-y)/h)/h^n$, $Q = Q(x, h)$. If $\mu = \nu =$ Lebesgue measure, we consider the maximal "approximate identity" operator $Mf(x) = \sup_{h>0}(1/h^n)\int\phi((x-y)/h) f(y)\,dy$ **[4]**. In this case $\lambda_{\phi_Q}(y) = |Q|\,|\{x: \phi(x) > y|Q|\}|$, and hence $\phi_Q^*(t) = \phi^*(t/|Q|)/|Q|$. Thus $\Phi(t) = \phi^*(t)$, and we get $(Mf)^*(\xi) \leq A\int_0^1 \phi^*(t) f^*(t\xi)\,dt$. This maximal inequality is due to Jurkat and Troutman **[4]** and our proof of Theorem 1 is a refinement of theirs.

3. Minkowski's integral inequality and Theorem 1 show that

$$(*)\qquad \|Mf\|_{p,\mu} \leq A\left(\int_0^\infty \frac{\Phi(t)}{t^{1/p}}\,dt\right)\|f\|_{p,\nu},$$

and hence $\int_0^\infty \Phi(t)/t^{1/p}\,dt < \infty$ implies that Mf is strong (p, p). In the setting of Lorentz spaces $L(p, q)$ **[2]**, this says that $\Phi \in L(p', 1)$, $1/p + 1/p' = 1$, implies strong (p, p) for Mf. A major part of this paper is devoted to the converse, i.e., when does strong (p, p) for Mf imply $\Phi \in L(p', 1)$? Simple examples show that this need not be the case in general. For, if we consider the "approximate identity" example of the previous section and assume that ϕ is radially nonincreasing, then $Mf(x) \leq \|\phi\|_1 M_0 f(x)$, where M_0 is the ordinary Hardy-Littlewood maximal operator. Simply take $\phi \in L^1$, $\phi \notin L(p', 1)$ to obtain an example.

We let now (u, v) be a pair of nonnegative functions (weights), i.e., $u \in L^1_{\text{loc}}$ and $0 < v < \infty$, a.e. x. This last restriction is made in order to avoid the special cases arising from division by zero, etc. Then

$$\frac{1}{|Q|}\int_Q f\,dx = \frac{1}{|Q|}\int f\cdot\frac{\chi_Q}{v} v\,dx = \int f\cdot\phi_Q\,d\nu,$$

where $\phi_Q(x) = \chi_Q(x)/|Q|v(x)$, and $d\nu = v\,dx$. If we let $d\mu = u\,dx$ and

$$\Phi(t) \equiv \Phi_{u,v}(t) = \sup\{\mu(Q)\phi_{Q,\nu}^*(\mu(Q)t)\},$$

then $\Phi \in L(p', 1)$ gives the double weight strong (p, p) for the ordinary Hardy-Littlewood maximal operator, which from now on we will denote by Mf.

4. The single weight problem, i.e., $u = v$, and the double weight problem are different and the Φ reflects this.

THEOREM 2. *Let* $1 < p < \infty$ *and* $\|Mf\|_{p,u} \leq A\|f\|_{p,v}$. *Then* $\Phi = \Phi_{u,v}$ *satisfies* (i) $\Phi(t) = O(t^{-1/p'})$, *as* $t \to 0$ *or* ∞, (ii) $\Phi(t) = O(t^a)$ *for* $0 > a > -1$ *as* $t \to \infty$.

PROOF. It is known that $(u, v) \in A_p$, i.e, $\int_Q u \cdot (\int_Q v^{1-p'})^{p-1} \leqslant c\,|Q|^p$ [5]. We note that

$$\left(\frac{\chi_Q}{v}\right)^*_\nu (\mu(Q)t) \leqslant \left[\frac{1}{\mu(Q)t}\int_0^{\mu(Q)t}\left(\frac{\chi_Q}{v}\right)^{*p'}_\nu (u)\,du\right]^{1/p'}$$
$$\leqslant \left[\frac{1}{\mu(Q)t}\int_Q \left(\frac{1}{v}\right)^{p'} v\,dx\right]^{1/p'}.$$

From this we get

$$\frac{\mu(Q)}{|Q|}\left(\frac{\chi_Q}{v}\right)^*_\nu (\mu(Q)t) \leqslant \frac{1}{t^{1/p'}}\,\frac{\mu(Q)^{1/p}}{|Q|}\cdot\left(\int_Q v^{1-p'}\right)^{(p-1)/p} \leqslant \frac{c}{t^{1/p'}},$$

and this proves (i).

For (ii) simply note that $\|Mf\|_{q,u} \leqslant A_q \|f\|_{q,v}$, $p \leqslant q$, so that by (i), $\Phi(t) = O(t^{-1/q'})$.

REMARK. (i) The above result shows that the behavior of Φ about 0 is much more critical than that about ∞. (ii) We have shown that $(u, v) \in A_p$ implies that $\Phi \in L(p', \infty)$.

THEOREM 3. *$\|Mf\|_{p,u} \leqslant A\|f\|_{p,u}$ for some $p > 1$ if and only if $\Phi \in L(p', 1)$, i.e., $\Phi \in L(p', 1)$ and $u \in A_p$ are equivalent.*

PROOF. Note that now $\phi_Q(x) = \chi_Q(x)/|Q|\,u(x)$, so that $\phi^*_{Q,\mu}(\mu(Q)t)$ is zero for $t > 1$, and hence $\Phi(t) = 0$, $t > 1$. Thus we have to show that $\int_0^1 \Phi(t)/t^{1/p}\,dt < \infty$. Since $u \in A_{p-\varepsilon}$ for some $\varepsilon > 0$ [5], we get from Theorem 2 that $\Phi \in L((p-\varepsilon)', \infty)$ from which $\Phi(t)/t^{1/p} \leqslant c/t^{1/(p-\varepsilon)'+1/p}$.

REMARK. Later we will show that $\Phi \in L(p', 1)$ implies $\Phi \in L((p-\varepsilon)', 1)$ in the single weight case without recourse to $u \in A_{p-\varepsilon}$.

5. From now on we assume that $n = 1$, and we will denote by I, J intervals in $\mathbf{R}$. In this section we will present a partial converse of Theorem 2, i.e., we ask whether $\Phi \in L(p', \infty)$ implies some norm inequality for Mf.

For Φ_1, Φ_2 two nonincreasing functions on $(0, \infty)$ we write $\Phi_1 \sim \Phi_2$ provided there are constants c_i, c_i', $i = 1, 2$, such that $c_1\Phi_1(c_1' t) \leqslant \Phi_2(t) \leqslant c_2\Phi_1(c_2' t)$, $0 < t \leqslant 1$.

THEOREM 4. *Let $\Phi_0 \geqslant 0$ be nonincreasing on $(0, \infty)$ such that $t\Phi_0(t)\downarrow 0$ as $t \downarrow 0$. Then $\Phi_0 \in L(p', \infty)$ on $[0, 1]$ if and only if there exists a pair of weights (u, v) such that $\Phi_{u,v} \sim \Phi_0$ and $\|Mf\|_{p,u} \leqslant A\|f\|_{p,v}$.*

REMARK. The condition $t\Phi_0(t)\downarrow 0$ as $t\downarrow 0$ can always be achieved by replacing Φ_0 by $\bar{\Phi}(t) = (1/t)\int_0^t \Phi_0 \geqslant \Phi_0(t)$ and $\bar{\Phi}$ is in the same $(p > 1)$ integrability class as Φ_0.

PROOF. By Theorem 2 we only need to show that $\Phi_0 \in L(p', \infty)$ implies the existence of (u, v). We may assume that $\Phi_0(1) = 1$ and $\Phi_0(t) \uparrow \infty$ as $t \downarrow 0$ (otherwise let $u = v = 1$). Let $\alpha_N = \Phi_0(2^{-N})$. Then $\alpha_N \leqslant A2^{N/p'}$, and since $2^{-N}\alpha_N \to 0$, we may assume that $\alpha_N \cdot 2^{-N} \leqslant \frac{1}{4}$, $N = 1, 2, \ldots$. Also note that $2^{-k}\alpha_{k+l} \leqslant \alpha_l$.

Let $J_N = [2^{N^2}, 2^{N^2} + \alpha_N 2^{-N}]$, $N = 1, 2, \ldots$, $J_0 = \mathbf{R} \setminus \bigcup_{N=1}^{\infty} J_N$, $K_N = [2^{N^2} + \frac{3}{4}, 2^{N^2} + 1]$. Define $v_N(t) = \alpha_N^{-1}$, $t \in J_N$, and $v_N(t) = 0$, $t \notin J_N$. Let $u_N(t) = 4$,

$t \in K_N$, and $u_N(t) = 0$, $t \notin K_N$. The desired pair of weights will be $v(t) = \Sigma v_N(t) + 4\chi_{J_0}(t)$, $u(t) = \Sigma u_N(t)$. We note that $\nu(J_N) = 2^{-N}$, from which $(\chi_{J_N})_\nu^*(t) = \chi_{[0,2^{-N}]}(t)$. Also $\mu(K_N) = 1$.

We wish to estimate

$$\Phi(t) \equiv \Phi_{u,v}(t) = \sup_I \frac{\mu(I)}{|I|}\left(\frac{\chi_I}{v}\right)_\nu^*(\mu(I)t),$$

and show that $\Phi \sim \Phi_0$. This will follow if for some constants c', c'', $c'\alpha_l \leq \Phi(2^{-l}) \leq c''\alpha_l$, $l = 1, 2, \ldots$. Our first observation is that

$$\left(\frac{\chi_I}{v}\right)_\nu^*(2^{-l}) \leq c\alpha_l, \qquad l = 1, 2, \ldots.$$

To see this, note that if $I \cap J_N = \varnothing$ for every N, then $(\chi_I/v)_\nu^*(t) \leq \frac{1}{4} \leq \alpha_l$. Otherwise, let $J_N, J_{N+1}, \ldots, J_M$ be all the J_i's with $J_i \cap I \neq \varnothing$. If $I_0 = [0, 2^{-M}]$, and for $j \geq 1$, $I_j = [2^{-M} + 2^{-M+1} + \cdots + 2^{-M+j-1}, 2^{-M} + 2^{-M+1} + \cdots + 2^{-M+j}]$, then $(\chi_I/v)_\nu^*(t) \leq \alpha_{M-j}$, $t \in I_j$. Thus, if $2^{-l} \in I_j$, $M - j \leq l + 1$, and since $2^{-N}\alpha_N \downarrow 0$, $2\alpha_l \geq \alpha_{l+1} \geq \alpha_{M-j}$.

It is clear that $\mu(I) \leq 4|I|$, and if $\mu(I) > 0$ and $I \cap J_N \neq \varnothing$ for some N, then $|I| \geq \frac{3}{4} - \alpha_N/2^N \geq \frac{1}{2}$. From this we see that, if $\mu(I) \geq 1$, then

$$\frac{\mu(I)}{|I|}\left(\frac{\chi_I}{V}\right)_\nu^*(\mu(I)2^{-l}) \leq c\left(\frac{\chi_I}{v}\right)_\nu^*(2^{-l}) \leq c\alpha_l,$$

and if $1/2^{k+1} \leq \mu(I) \leq 1/2^k$, and $I \cap J_N \neq \varnothing$ for some N, then

$$\frac{\mu(I)}{|I|}\left(\frac{\chi_I}{v}\right)_\nu^*(\mu(I)2^{-l}) \leq c2^{-k}\left(\frac{\chi_I}{v}\right)_\nu^*(2^{-k-l-1}) \leq c2^{-k}\alpha_{k+l+1} \leq c\alpha_l.$$

This shows that $\Phi(2^{-l}) \leq c\alpha_l$, and since for $I = [2^{l^2}, 2^{l^2} + 1]$,

$$\frac{\mu(I)}{|I|}\left(\frac{\chi_I}{v}\right)_\nu^*(\mu(I)2^{-l}) = \alpha_l, \qquad \alpha_l \leq \Phi(2^{-l}).$$

We proceed now with the proof of $\|Mf\|_{p,u} \leq A\|f\|_{p,v}$. Let $f \geq 0$, $f_N = f\chi_{J_N}$, $N = 0, 1, \ldots$. Then

$$\begin{aligned}\int (Mf)^p u\,dx &= \int \left[M\left(f_0 + \sum f_N\right)\right]^p u\,dx \\ &\leq 2^{p-1}\int (Mf_0)^p u\,dx + 2^{p-1}\int \left[M\left(\sum f_N\right)\right]^p u\,dx.\end{aligned}$$

Note that $\int (Mf_0)^p u\,dx \leq c\int (Mf_0)^p\,dx \leq c\|f_0\|_p^p \leq c\|f\|_{p,v}^p$.

We next claim that $\int (Mf_N)^p u_N\,dx \leq A\int f_N^p v_N\,dx$. For $x \in K_N = \operatorname{supp} u_N$ we have

$$(Mf_N)^p(x) \leq \left(2\int_{2^{N^2}}^{2^{N^2}+\alpha_N 2^{-N}} f_N\,dx\right)^p \leq 2^p\left(\alpha_N 2^{-N}\right)^{p/p'}\|f_N\|_p^p.$$

Since $\alpha_N = O(2^{N/p'})$ we obtain $(\alpha_N 2^{-N})^{p/p'} \leq c\alpha_N^{-1}$ from which $(Mf_N)^p \leq c(1/\alpha_N)\|f_N\|_p^p = c\int f_N^p v_N\,dx$.

We next observe that $\int M^p(\Sigma f_N)u\,dx = \Sigma_k \int M^p(\Sigma f_N)u_k\,dx$, and thus, using $(\Sigma a_j)^p \leqslant \Sigma 2^{j(p-1)}a_j^p$, we get

$$\int M^p\Big(\sum f_N\Big)u_k\,dx \leqslant 2^{p-1}\sum_{N\geqslant k} 2^{(N-k)(p-1)}\int M^p f_N u_k\,dx + 2^{p-1}\sum_{N<k} 2^{(k-N)(p-1)}\int M^p f_N u_k\,dx.$$

For $k < N$ we get

$$\int M^p f_N u_k\,dx \leqslant c\int\left(\frac{1}{2^{N^2}-2^{k^2}}\int_{J_N} f_N\right)^p u_k(x)\,dx \leqslant \frac{c}{2^{pN^2}}\left(\int_{J_N} f_N\right)^p \int u_k\,dx \leqslant \frac{c}{2^{pN^2}}\int M^p f_N u_N\,dx,$$

since $\int_{\mathbf{R}} u_k\,dx = \int_{\mathbf{R}} u_N\,dx$.

Similarly, if $k > N$, $\int M^p f_N u_k\,dx \leqslant (c/2^{pk^2})\int M^p f_N u_N\,dx$, and thus we get

$$\int M^p\Big(\sum f_N\Big)u_k\,dx \leqslant 2^{p-1}\int M^p f_k\cdot u_k\,dx + c2^{p-1}\left\{\sum_{N>k}\frac{2^{N(p-1)}}{2^{pN^2}\cdot 2^{k(p-1)}}\int M^p f_N\cdot u_N\,dx + \sum_{N<k}\frac{2^{k(p-1)}}{2^{pk^2}2^{N(p-1)}}\int M^p f_N\cdot u_N\,dx\right\}.$$

We sum this over k and interchange the order of summation to get

$$\int M^p f u\,dx \leqslant 2^{(p-1)}\sum_{k=1}^{\infty}\int M^p f_k u_k\,dx + c2^{p-1}\left\{\sum_{N=1}^{\infty}\sum_{k=1}^{N}\frac{2^{N(p-1)}}{2^{pN^2}2^{k(p-1)}}\int M^p f_N u_N\,dx + \sum_{N=1}^{\infty}\sum_{k=N}^{\infty}\frac{2^{k(p-1)}}{2^{pk^2}\cdot 2^{N(p-1)}}\int M^p f_N u_N\,dx\right\}$$
$$\leqslant A\sum_{k=1}^{\infty}\int M^p f_k u_k\,dx \leqslant A\sum_{k=1}^{\infty}\int f_k^p v_k\,dx \leqslant A\int f^p v\,dx.$$

REMARK. Under the hypothesis of Theorem 4, $\Phi_0 \in L(p', \infty)$ if and only if there exists (u, v) for which $\mu\{x\colon Mf(x) > y\} \leqslant c\|f\|_{p,v}^p/y^p$ and $\Phi_{u,v} \sim \Phi_0$.

6. In order to make a more detailed study of $\Phi = \Phi_{u,v}$ for a pair of weights (u, v) we need some preliminary results. Again our analysis will take place on $\mathbf{R}$.

For $f\colon \mathbf{R} \to [0, \infty]$, and I, J compact intervals let

$$M_{j,I}f(x) = \sup_{x\in J\subset I}\frac{1}{|J|}\int_J M_{j-1,I}f(y)\,dy,$$

the jth iterated maximal function relative to I. Set $M_{j,I}f(x) = 0$, $x \notin I$. $M_{0,I}f(x) = f(x)\chi_I(x)$.

LEMMA 1. *Let $f \geqslant 0$ be in $L^1(I)$, $\operatorname{supp} f \subset I$, and let $g \geqslant 0$ be in $L^1(J)$, $\operatorname{supp} g \subset J$, and assume $|I| = |J|$. Assume there are constants $C \geqslant 1$, $A \geqslant 1$ with*

$$C|\{x \in I: f(x) > \alpha/A\}| \geqslant |\{y \in J: g(y) > \alpha\}|, \qquad \alpha > 0.$$

Then there is a constant B so that

$$ACB^j \int_I M_{j,I} f \geqslant \int_J M_{j,J} g.$$

PROOF. We will first establish that for $\alpha > 0$

$$(1) \qquad 2BC\left|\left\{x \in I: M_{1,I}f(x) > \frac{\alpha}{2A}\right\}\right| \geqslant |\{y \in J: M_{1,J}g(y) > \alpha\}|,$$

where B is the Besicovitch covering constant. To do this, we may assume that $\alpha/2A \geqslant (1/|I|)\int_I f$, as otherwise $M_{1,I}f(x) > \alpha/2A$, $x \in I$, and (1) follows. We have

$$|\{y \in J: M_{1,J}g(y) > \alpha\}| \leqslant \frac{2B}{\alpha}\int_{\{g > \alpha/2\}} g = \frac{2B}{\alpha}\left[\frac{\alpha}{2}\lambda_g\left(\frac{\alpha}{2}\right) + \int_{\alpha/s}^{\infty} \lambda_g(\tau)\, d\tau\right],$$

where $\lambda_g(\tau) = |\{x \in J: g(x) > \tau\}|$. By hypothesis this is majorized by

$$\frac{2BC}{\alpha}\left[\frac{\alpha}{2}\lambda_f\left(\frac{\alpha}{2A}\right) + \int_{\alpha/2}^{\infty} \lambda_f\left(\frac{\tau}{A}\right) d\tau\right] = \frac{2BCA}{\alpha}\int_{\{f > \alpha/2A\}} f$$

$$\leqslant 2BC\left|\left\{x \in I: M_{1,I}f(x) > \frac{\alpha}{2A}\right\}\right|$$

(see [**8**, p. 23]).

We now iterate (1) and get

$$(2B)^j C\left|\left\{x \in I: M_{j,I}f(x) > \frac{\alpha}{2^j A}\right\}\right| \geqslant \left|\{y \in J: M_{j,J}g(y) > \alpha\}\right|.$$

Consequently, $\int_J M_{j,J} g = \int_0^\infty \lambda_{M_{j,J}g}(\alpha)\, d\alpha \leqslant (4B)^j CA \int_I M_{j,I} f$.

LEMMA 2. *Let $(u, v) \in A_p$ for some $p > 1$, i.e., $\int_I u \cdot (\int_I v^{1-p'})^{p-1} \leqslant c|I|^p$, and form $\Phi = \Phi_{u,v}$. Assume that $|v^{-1}(t)| = 0$, $t > 0$. Then for each N there exists α_N and compact intervals $I_N \supset J_N$, $I_N \supset I'_N$, such that $J_N \cap I'_N = \varnothing$ and J_N, I'_N have an endpoint in common with I_N, and there is $S_N \subset J_N$ such that*

(i) $\Phi(2^{-N}) \leqslant c\mu(I'_N)\alpha_N/|I_N| \leqslant c2^{N/p'}$,

(ii) $\alpha_N \leqslant 1/v(x) \leqslant 5\alpha_N$, $x \in S_N$,

(iii) $\mu(I'_N)/(5 \cdot 2^N) \leqslant \nu(S_N) \leqslant \mu(I'_N)/2^N$,

(iv) $\alpha_N \leqslant (\chi_{S_N}/v)^*_\nu(\mu(I'_N)/(5 \cdot 2^N)) \leqslant 5\alpha_N$.

PROOF. Since

$$\Phi(2^{-N}) = \sup_I \frac{\mu(I)}{|I|}\left(\frac{\chi_I}{v}\right)^*_\nu\left(\frac{\mu(I)}{2^N}\right)$$

choose an interval $\bar{I}_N = [a_N, b_N]$ for which

$$\Phi(2^{-N}) \leqslant 2\frac{\mu(\bar{I}_N)}{|\bar{I}_N|}\left(\frac{\chi_{\bar{I}_N}}{v}\right)^*_\nu(\mu(\bar{I}_N)2^{-N}).$$

We can pick points $a_N = x_0 < x_1 < x_2 < x_3 < x_4 = b_N$ for which $\int_{x_{i-1}}^{x_i} u\,dx = \mu(\bar{I}_N)/4$, $i = 1, 2, 3, 4$. If $I_{N,i} = [x_{i-1}, x_i]$, we have, since

$$\left(\frac{\chi_{\bar{I}_N}}{v}\right)_\nu^*(\tau) \leq \sum_{i=1}^{4}\left(\frac{\chi_{I_{N,i}}}{v}\right)_\nu^*\left(\frac{\tau}{4}\right),$$

an i such that for each j,

$$\Phi(2^{-N}) \leq c\frac{\mu(I_{N,j})}{|\bar{I}_N|}\left(\frac{\chi_{I_{N,i}}}{v}\right)_\nu^*\left(\frac{\mu(I_{N,j})}{2^N}\right).$$

Select now a j so that $I_{N,j} \cap I_{N,i} = \varnothing$, and write $J_N = I_{N,i}$, $J_N^* = I_{N,j}$ and let I_N be the smallest interval containing $J_N \cup J_N^*$. Then

$$\Phi(2^{-N}) \leq c\frac{\mu(J_N^*)}{|I_N|}\left(\frac{\chi_{J_N}}{v}\right)_\nu^*\left(\frac{\mu(J_N^*)}{2^N}\right).$$

Let us denote by J' an interval in J_N^* which has that endpoint in common with J_N^* which J_N^* has in common with I_N, and set

$$\bar{\Phi}(2^{-N}) = \sup_{J'}\frac{\mu(J')}{|I_N|}\left(\frac{\chi_{J_N}}{v}\right)_\nu^*\left(\frac{\mu(J')}{2^N}\right).$$

Select now J' for which the sup is "attained", i.e.

$$\bar{\Phi}(2^{-N}) \leq 2\frac{\mu(J')}{|I_N|}\left(\frac{\chi_{J_N}}{v}\right)_\nu^*\left(\frac{\mu(J')}{2^N}\right),$$

and let $\alpha_N = (\chi_{J_N}/v)_\nu^*(\mu(J')/2^N)$.

We define $S_N \in \{x \in J_N\colon 5\alpha_N \geq 1/v(x) \geq \alpha_N\}$, and $S_N' = \{x \in J_N\colon 1/v(x) \geq \alpha_N\}$. Since $|v^{-1}(t)| = 0$, $t > 0$, we see that $\nu(S_N') = \nu\{x \in J_N\colon 1/v(x) > \alpha_N\} = \mu(J')/2^N$.

We claim now that $\nu(S_N) \geq \frac{1}{5}\nu(S_N')$. To prove this we may assume that $\nu(S_N') > \nu(S_N)$. If $\nu(S_N) < \frac{1}{5}\nu(S_N')$, then

$$\frac{\mu(J')}{2^N} \geq \nu(S_N' \setminus S_N) > \frac{4}{5}\nu(S_N') = \frac{4}{5}\frac{\mu(J')}{2^N}.$$

We can now choose an interval $J'' \subset J_N^*$ for which $\mu(J'')2^{-N} \leq \nu(S_N' \setminus S_N) \leq \mu(J'')2^{-N+1}$, and J'' is a candidate for the sup of $\bar{\Phi}$. Then $\mu(J'') > \frac{2}{5}\mu(J')$, and since $(\chi_{S_N' \setminus S_N}/v)_\nu^*(\mu(J'')/2^N) \geq 5\alpha_N$ we get

$$\bar{\Phi}(2^{-N}) \geq \frac{\mu(J'')}{|I_N|}\left(\frac{\chi_{S_N' \setminus S_N}}{v}\right)_\nu^*\left(\frac{\mu(J'')}{2^N}\right) > 2\alpha_N\frac{\mu(J')}{|I_N|} \geq \bar{\Phi}(2^{-N}).$$

Hence $\frac{1}{5}\mu(J')/2^N \leq \nu(S_N) \leq \mu(J')/2^N$.

If we let $I_N' = J'$, the properties (ii), (iii), and (iv) of the lemma follow, and the only thing that remains is $\mu(I_N')\alpha_N/|I_N| \leq c2^{N/p'}$. This can be done by the same argument used in Theorem 2 for (i) since $(u, v) \in A_p$.

7. It is well known that $u \in A_p$, $p > 1$, implies that $u \in A_{p-\varepsilon}$ for some $\varepsilon > 0$, and that this is no longer the case for $(u, v) \in A_p$ [**5, 6**]. If we want $(u, v) \in A_{p-\varepsilon}$, then

in terms of $\Phi = \Phi_{u,v}$ we need to prove that $\Phi \in L((p-\varepsilon)', 1)$. Here is where the behavior of the iterated maximal operator $M_j f$ comes into the picture.

THEOREM 5. *Let $(u, v) \in A_p$, $1 < p$, and let $\Phi = \Phi_{u,v}$. Assume that $\| M_j f \|_{p,u} \leqslant A_j \| f \|_{p,v}$, $f \in L_v^p$, $j = 1, 2, \ldots$. Then there are constants $c > 0$, $B > 0$ such that for every j, N,*

$$\Phi(2^{-N}) \leqslant c \frac{A_{j+1}}{B^j} \left(\frac{j!}{N^j} \right) 2^{N/p'}.$$

PROOF. We will first show that we may assume that $|v^{-1}(t)| = 0$, $t > 0$. Since our overall assumption on v is $0 < v < \infty$ a.e., we choose $v(x) \leqslant \bar{v}(x) \leqslant 2v(x)$ such that $|\bar{v}^{-1}(t)| = 0$, $t > 0$. Then $(u, \bar{v}) \in A_p$ and $\Phi_{u,v}(t) \leqslant 2\Phi_{u,\bar{v}}(t)$.

We now choose $I_N \supset J_N, I'_N$, $S_N \subset J_N$, and α_N as in Lemma 2. Then

$$\alpha_N \approx \left(\frac{\chi_{S_N}}{v} \right)^*_{\nu} \left(\frac{\mu(I'_N)}{5 \cdot 2^N} \right)$$

and

(i) $\alpha_N \leqslant 1/v(x) \leqslant 5\alpha_N$, $x \in S_N$,
(ii) $|S_N|/5\alpha_N \leqslant \nu(S_N) \leqslant |S_N|/\alpha_N$,
(iii) $\mu(I'_N)/(5 \cdot 2^N) \leqslant \nu(S_N) \leqslant \mu(I'_N)/2^N$,
(iv) $\Phi(2^{-N}) \leqslant c\mu(I'_N)\alpha_N/|I_N| \leqslant c \cdot 2^{N/p'}$.

We begin with

$$\int_{I'_N} \left\{ M_{j+1}\left(v^{1-p'}\chi_{S_N}\right) \right\}^p u\, dx \geqslant \frac{\mu(I'_N)}{|I_N|^p} \left\{ \int_{I_N} M_j\left(v^{1-p'}\chi_{S_N}\right) \right\}^p$$

$$\geqslant \frac{\mu(I'_N)}{|I_N|^p} \left\{ \int_{I_N} M_{j,I_N}\left(\alpha_N^{p'-1}\chi_{S_N}\right) \right\}^p.$$

By Lemma 1 this is

$$\geqslant B^{jp} \frac{\alpha_N^{(p'-1)p}\mu(I'_N)}{|I_N|^p} \left(\int_{|S_N|}^{|I_N|} M_{j,H_N}\left(\chi_{[0,|S_N|]}\right) dx \right)^p,$$

where $H_N = [0, |I_N|]$.

Since for $|S_N| \leqslant t_1 \leqslant |I_N|$, $M_{1,H_N}(\chi_{[0,|S_N|]})(t_1) \geqslant |S_N|/t_1$ we see that

$$\int_{|S_n|}^{|I_N|} M_{2,H_N}\left(\chi_{[0,|S_N|]}\right) \geqslant \int_{|S_N|}^{|I_N|} \frac{1}{t_2} \int_{|S_N|}^{t_2} M_{1,H_N}\left(\chi_{[0,|S_N|]}\right) dt_1$$

$$\geqslant \int_{|S_N|}^{|I_N|} \frac{|S_N|}{t_2} \log\left(\frac{t_2}{|S_N|} \right) dt_2 = \frac{|S_N|}{2} \log^2 \frac{|I_N|}{|S_N|}.$$

Thus in general,

$$\int_{I'_N} \left\{ M_{j+1}\left(v^{1-p'}\chi_{S_N}\right) \right\}^p u\, dx \geqslant \frac{\alpha_N^{(p'-1)p}\mu(I'_N)}{|I_N|^p} |S_N|^p \left[\frac{B^j \log^j\left(\frac{|I_N|}{|S_N|} \right)}{j!} \right]^p.$$

Since $\| M_{j+1} f \|_{p,u} \leqslant A_{j+1} \| f \|_{p,v}$ we get with $f = v^{1-p'} \chi_{S_N}$,

$$\frac{\alpha_N^{(p'-1)p}}{|I_N|^p} \mu(I_N') |S_N|^p \left[\frac{B^j \log^j\left(\frac{|I_N|}{|S_N|}\right)}{j!} \right]^p$$

$$\leqslant A_{j+1}^p \int_{S_N} v^{1-p'} dx \leqslant 5^{p'-1} A_{j+1}^p \alpha_N^{p'-1} |S_N|,$$

or

$$\frac{\alpha_N \mu(I_N')}{|I_N|^p} |S_N|^{p-1} \leqslant c \left[\frac{A_{j+1} \cdot j!}{B^j \log^j\left(\frac{|I_N|}{|S_N|}\right)} \right]^p.$$

Since $|S_N| \leqslant 5\alpha_N \nu(S_N) \leqslant 5\alpha_N \mu(I_N')/2^N \leqslant c|I_N|/2^N \cdot 2^{N/p'} = c|I_N|/2^{N/p}$, we get $|I_N|/|S_N| \geqslant c 2^{N/p}$ from which

$$\frac{\alpha_N \mu(I_N')}{|I_N|^p} \left[\frac{\alpha_N}{5} \frac{\mu(I_N')}{2^N} \right]^{p-1} \leqslant c \left[\frac{A_{j+1} \cdot j!}{B^j \log^j(c 2^{N/p})} \right]^p.$$

From this we finally obtain

$$\frac{\alpha_N \mu(I_N')}{|I_N|} \leqslant c \frac{A_{j+1} \cdot j!}{B^j N^j} \cdot 2^{N/p'},$$

and the proof is complete.

We can replace in Theorem 5 the strong (p, p) for $M_j f$ by weak (p, p) and obtain the same result. We state this as

Corollary. *If* $(u, v) \in A_p$, $1 < p$, $\Phi = \Phi_{u,v}$, *and* $\mu\{x\colon M_j f(x) > y\} \leqslant A_j \| f \|_{p,v}^p / y^p$, $j = 1, 2, \ldots$, *then there are constants* $c > 0$, $B > 0$ *such that for every* j, N,

$$\Phi(2^{-N}) \leqslant c \frac{A_{j+1}}{B^j} \left(\frac{j!}{N^j} \right) 2^{N/p'}.$$

Proof. Start out exactly as in Theorem 5, and note that for $x \in I_N'$,

$$\left\{ M_{j+1}\left(v^{1-p'} \chi_{S_N}\right) \right\}^p(x) \geqslant \left\{ \frac{1}{|I_N|^p} \int_{I_N} M_j\left(v^{1-p'} \chi_{S_N}\right) \right\}^p.$$

If we let y^p be the right side of this inequality, then

$$y^p \mu\left\{ x\colon M_{j+1}\left(v^{1-p'} \chi_{S_N}\right) > y \right\} \geqslant \frac{\mu(I_N')}{|I_N|^p} \left[\int_{I_N} M_j\left(v^{1-p'} \chi_{S_N}\right) \right]^p.$$

The rest of the proof is exactly as that of Theorem 5.

Theorem 6. *Let* $(u, v) \in A_p$ *for some* $p > 1$, *and form* $\Phi = \Phi_{u,v}$.

(i) *If* $\mu\{x\colon M_3 f(x) > y\} \leqslant A \| f \|_{p,v}^p / y^p$, *then* $\Phi \in L(p', 1)$ *and hence* $\| Mf \|_{p,u} \leqslant A \| f \|_{p,v}$

(ii) *If* $\sup_{\|f\|_{p,v}=1} \|M_j f\|_{p,u} = o(A^j)$, *then there exists* $\varepsilon > 0$ *such that* $\Phi \in L((p-\varepsilon)', 1)$, *and hence* $\|Mf\|_{p-\varepsilon,u} \leq A\|f\|_{p-\varepsilon,v}$.

PROOF. To prove (i) we use the corollary and obtain $\Phi(2^{-N}) \leq c2^{N/p'}/N^2$ and so $\Sigma\Phi(2^{-N})/2^{N/p'} < \infty$. Hence $\Phi \in L(p', 1)$.

For (ii) we use Theorem 5 and get $\Phi(2^{-N}) \leq c(A/N)^j j! 2^{N/p'}$, for some constant A. Since by Stirling's formula $j! \sim \sqrt{2\pi}\, e^{-j} j^{j+1/2}$, we get

$$\Phi(2^{-N}) \leq c\left(\frac{Aj}{eN}\right)^j j^{1/2} \cdot 2^{N/p'}.$$

If we now let $\alpha = e/2A$ and $j = [\alpha N]$, then

$$\Phi(2^{-N}) \leq c\frac{N^{1/2}}{2^{\alpha N}} 2^{N/p'} \leq \frac{c}{N^2} 2^{N/p' - \alpha N/2} \leq \frac{c}{N^2} 2^{N/(p-\varepsilon)'},$$

for some $\varepsilon > 0$. Thus $\Sigma\Phi(2^{-N})/2^{N/(p-\varepsilon)'} < \infty$ and so $\Phi \in L((p-\varepsilon)', 1)$.

REMARK. Theorem 6 provides us with a different proof of $u \in A_p$ implies $u \in A_{p-\varepsilon}$ for some $\varepsilon > 0$. From **[7, 3]** we know that $u \in A_p$ implies $\|Mf\|_{p,u} \leq A\|f\|_{p,u}$ without recourse to $A_{p-\varepsilon}$. But then $\|M_j f\|_{p,u} \leq A^j \|f\|_{p,u}$, and thus from (ii), $\|Mf\|_{p-\varepsilon,u} \leq B\|f\|_{p-\varepsilon,u}$ from which we get $u \in A_{p-\varepsilon}$.

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